Writing on the Facade of RWTH ICT Cubes: Cost Constrained Geometric Huffman Coding

Georg Böcherer, Fabian Altenbach, Martina Malsbender*,
and Rudolf Mathar

Institute for Theoretical Information Technology
*kadawittfeldarchitektur

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RWTH Information and Communication Technology Cubes

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Slats

Slats keep building from heating up.

- 3 types of slats: left, right, and middle.
- height is 1.7m
- widths $\mathbf{w} = (0.18m,\ 0.18m,\ 0.31m)^T$.
- slats are placed each 0.625m.

Design constraints

C1. For aesthetic reasons, the sequence of slats should appear random.

C2. To ensure enough cooling, around 33% of the facade area should be covered by the slats.

C3. Since shadow turns the rooms dark, the total shadowing should not exceed 33%.

Probabilistic Model

Random slat sequence:

- slats iid $\sim \mathbf{p}$.
- average width $\mathbf{w}^T\mathbf{p}$.

Some notation:

- $\mathbf{u} := (1/3,\ 1/3,\ 1/3)^T$ denotes uniform distribution.
- Kullback-Leibler distance of two pmfs $\mathbf{p}, \mathbf{q}$ is

$$\mathbb{D}(\mathbf{p}\|\mathbf{q}) = \sum_i p_i \log \frac{p_i}{q_i}.$$

Turn Design Constraints into Mathematical Problem

C1. For aesthetic reasons, the sequence of slats should appear random. minimize$_{\mathbf{p}}\, \mathbb{D}(\mathbf{p} \| \mathbf{u})$

C2. To ensure enough cooling, around 33% of the facade area should be covered by the slats.

C3. Since shadow turns the rooms dark, the total shadowing should not exceed 33%. subject to $\mathbf{w}^T\mathbf{p} \leq S = 0.2063$

(Since $\mathbf{w}^T\mathbf{u} > S$, constraint C3. is active and constraint C2. can be dropped.)

- Solution is $\mathbf{p}^* = (0.3988,\ 0.3988,\ 0.2023)^T$
- Resulting shadowing is 33%.

Idea: Write a Text to the Facade

Map a text to the slats such that the resulting slat sequence fulfills the design constraints.

Approach: Source-Channel Separation

- Source encoder: map the text to a sequence of equiprobable bits [1].
- Channel encoder: map the binary sequence to a sequence of slats (this work).

[1] F. Altenbach, G. Böcherer, R. Mathar "Short Huffman Codes Producing 1s Half of the Time," to be presented at ICSPCS 2011, Honolulu.

Prefix-free matcher
Example

- Slats $\{\ell, r, m\}$, full prefix-free code $\{1, 00, 01\}$
- Prefix-free matcher: $$\begin{aligned} 1 &\mapsto \ell \\ 01 &\mapsto r \\ 00 &\mapsto m \end{aligned}$$
- Matching: $10001101\cdots \to \ell m r \ell r \cdots$
- Generated distribution $\mathbf{d}$:

$$d_\ell = 2^{-1}, \qquad d_r = 2^{-2}, \qquad d_m = 2^{-2}$$

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Dyadic pmfs and Full Prefix-Free Codes

- $\mathbf{d}$ is a dyadic pmf if

$$\sum_i d_i = 1$$

$$\text{for } i = 1, \ldots, n : \exists \ell_i \in \mathbf{N} : d_i = 2^{-\ell_i}.$$

- A prefix-free code with codeword lengths $\boldsymbol{\ell}$ is a full prefix-free code if

$$\sum_i 2^{-\ell_i} = 1.$$

- Every dyadic pmf can be generated by a full prefix-free code.
- Every full prefix-free code generates a dyadic pmf.

Optimization problem

$$\begin{aligned} \underset{\text{dyadic } \mathbf{d}}{\text{minimize}} \quad & \mathbb{D}(\mathbf{d} \| \mathbf{u}) \\ \text{subject to} \quad & \mathbf{w}^T \mathbf{d} \leq S. \end{aligned}$$

Geometric Huffman coding

- Without the constraint, the problem is optimally solved by geometric Huffman coding (GHC) [2,3].
- For a target pmf $\mathbf{p}$ with $p_1 \geq p_2 \geq \cdots \geq p_n$, GHC recursively constructs a full prefix-free code with the updating rule

$$p' = \begin{cases} 2\sqrt{p_n p_{n-1}} & \text{if } p_{n-1} < 4p_n \\ p_{n-1} & \text{if } p_{n-1} \geq 4p_n. \end{cases}$$

- The induced dyadic pmf minimizes $\mathbb{D}(\mathbf{d}\|\mathbf{p})$ over all dyadic pmfs $\mathbf{d}$.

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[2] G. Böcherer and R. Mathar, "Matching Dyadic Distributions to Channels," presented at DCC 2011, Snowbird.

[3] www.georg-boecherer.de/ghc

Cost Constrained Geometric Huffman coding (ccGhc)

- ccGhc adds a scaled version of the constraint to the objective function:

$$\mathbb{D}(\mathbf{d}\|\mathbf{p}) + \lambda \mathbf{w}^T \mathbf{p} = \mathbb{D}(\mathbf{d}\|\mathbf{p} \circ e^{-\lambda \mathbf{w}}).$$

  The right-hand side is minimized by applying Ghc to $\mathbf{p} \circ e^{-\lambda \mathbf{w}}$.

- The best value of λ is found via bisection.

Asymptotic optimality

Proposition: Jointly generate k consecutive slats. Let $k \to \infty$.

- The per slat Kullback-Leibler distance converges to the optimal value $\mathbb{D}(\mathbf{p}^* \| \mathbf{u})$.
- The per slat average width converges from below to the optimal value S.

Code found by ccGhc

Jointly generating $k = 3$ slats turned out to be a good choice.

0010 : $\ell\ell\ell$	1101 : $\ell\ell r$	00000 : $\ell\ell m$
1100 : $\ell r\ell$	1111 : ℓrr	00011 : ℓrm
00010 : $\ell m\ell$	01101 : ℓmr	0000111 : ℓmm
1110 : $r\ell\ell$	1001 : $r\ell r$	01100 : $r\ell m$
1000 : $rr\ell$	1011 : rrr	01111 : rrm
01110 : $rm\ell$	01001 : rmr	000010 : rmm
01000 : $m\ell\ell$	01011 : $m\ell r$	001101 : $m\ell m$
01010 : $mr\ell$	1010 : mrr	001100 : mrm
001111 : $mm\ell$	001110 : mmr	0000110 : mmm

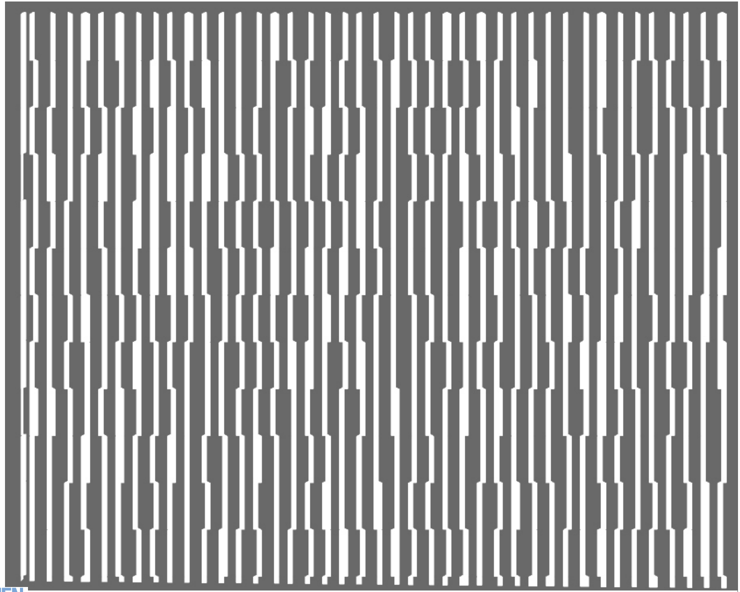
Numerical Results

Applying the code to the compressed text results in

- $$\mathbf{p}_{\text{eff}} = \frac{1}{4264}(\sharp\{\textit{left}\}, \sharp\{\textit{right}\}, \sharp\{\textit{middle}\})^T$$
  $$= (0.3875,\ 0.4089,\ 0.2036)^T.$$

  The optimal pmf is $\mathbf{p}^* = (0.3988,\ 0.3988,\ 0.2023)^T$.
- The resulting shadowing is 33.03%.

A written facade



Discrete Memoryless Channel with Power Constraint

- $\mathbf{p}^*$ capacity achieving pmf of a DMC with symbol powers $\mathbf{w}$ and power constraint E.
- The optimal prefix-free matcher is given by the solution of

$$\begin{aligned} \underset{\text{dyadic } \mathbf{d}}{\text{minimize}} \quad & \mathbb{D}(\mathbf{d} \| \mathbf{p}^*) \\ \text{subject to} \quad & \mathbf{w}^T \mathbf{d} \leq E. \end{aligned}$$

- ccGhc asymptotically solves this problem. This fixes the proof given in [4]. See [5] for details.

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[4] G. Böcherer, F. Altenbach, R. Mathar "Capacity-achieving modulation for fixed constellations with average power constraint," presented at ICC 2011, Kyoto.

[5] G. Böcherer, "Capacity-achieving probabilistic shaping for noisy and noiseless channels," submitted as dissertation.